ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Sparse stochastic processes

Chapter 3
Mathematical context and background

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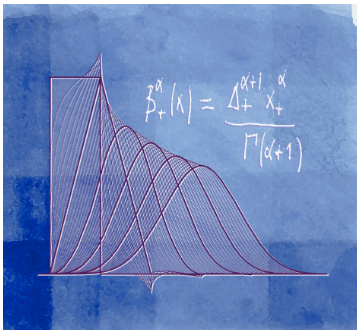


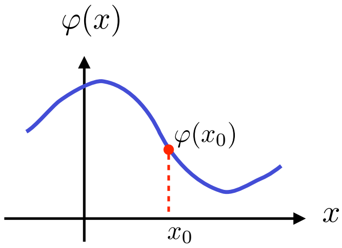
February 2017

EDEE Course

CONTENT

- **3.1 Some functions spaces**
  - Banach spaces
  - Hilbert spaces
  - Nuclear-Fréchet spaces
- **3.2 Dual spaces and adjoint operators**
- **3.3 Generalized functions**
  - Generalized Fourier transform
  - The kernel theorem
  - LSI operators and convolution

On the notation



Notion of function: conventional vs. abstract interpretation

- Map $\varphi : \mathbb{R}^d \to \mathbb{R}$ (or $\mathbb{C}$) such that $\boldsymbol{x} \mapsto \varphi(\boldsymbol{x})$
- Element of a vector space: $\varphi \in \mathcal{X}$

Generic time or space indices: t, $\boldsymbol{x} = (x_1, \cdots, x_d)$ or $\boldsymbol{r} = (r_1, \cdots, r_d)$

Equivalent notations: $\varphi, \varphi(\cdot), \varphi(\boldsymbol{x})$

Specific sample values $\varphi(\boldsymbol{x}_0), s(\boldsymbol{x}_1) \in \mathbb{R}$

vs. (continuous) linear functionals $\varphi \mapsto \langle f, \varphi \rangle \in \mathbb{R}$

Sampling property of Dirac distribution: $s(\boldsymbol{x}_0) = \langle \delta(\cdot - \boldsymbol{x}_0), s \rangle$

(Hypothesis: s is bounded and continuous)

Sets

$\mathbb{N}, \mathbb{Z}^+$	Non-negative integers, including 0
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Non-negative real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{Z}^d$	d-dimensional integers

Various notations

j	Imaginary unit such that $\mathrm{j}^2 = -1$
$\lceil x \rceil$	Ceiling: smallest integer at least as large as x
$\lfloor x \rfloor$	Floor: largest integer not exceeding x
$(x_1 : x_n)$	n-tuple $(x_1, x_2, \dots, x_n)$
$\\|f\\|$	Norm of the function f (see Section 3.1.2)
$\\|f\\|_{L_p}$	L_p-norm of the function f (in the sense of Lebesgue)
$\\|a\\|_{\ell_p}$	ℓ_p-norm of the sequence a
$\langle \varphi, s \rangle$	Scalar (or duality) product
$\langle f, g \rangle_{L_2}$	L_2 inner product
$f^\vee$	Reversed signal: $f^\vee(\boldsymbol{r}) = f(-\boldsymbol{r})$
$(f * g)(\boldsymbol{r})$	Continuous-domain convolution
$(a * b)[\boldsymbol{n}]$	Discrete-domain convolution
$\hat{\varphi}(\boldsymbol{\omega})$	Fourier transform of φ: $\int_{\mathbb{R}^d} \varphi(\boldsymbol{r}) \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{\omega}, \boldsymbol{r} \rangle} \, \mathrm{d}\boldsymbol{r}$
$\hat{f} = \mathscr{F}\{f\}$	Fourier transform of f (classical or generalized)
$f = \mathscr{F}^{-1}\{\hat{f}\}$	Inverse Fourier transform of $\hat{f}$
$\overline{\mathscr{F}}\{f\}(\boldsymbol{\omega}) = \mathscr{F}\{f\}(-\boldsymbol{\omega})$	Conjugate Fourier transform of f

Signals, functions, and kernels

$f, f(\cdot)$, or $f(\boldsymbol{r})$	Continuous-domain signal: function $\mathbb{R}^d \to \mathbb{R}$
φ	Generic test function in $\mathscr{S}(\mathbb{R}^d)$
$\psi_{\mathrm{L}} = \mathrm{L}^* \phi$	Operator-like wavelet with smoothing kernel ϕ
s, $\langle \varphi, s \rangle$	Generalized function $\mathscr{S}(\mathbb{R}^d) \to \mathbb{R}$
μ_h	Measure associated with h: $\langle \varphi, h \rangle = \int_{\mathbb{R}^d} \varphi(\boldsymbol{r}) \mu_h(\mathrm{d}\boldsymbol{r})$
δ	Dirac impulse: $\langle \varphi, \delta \rangle = \varphi(\mathbf{0})$
$\delta(\cdot - \boldsymbol{r}_0)$	Shifted Dirac impulse

From linear algebra to functional analysis

	Infinite-dim counterpart
$\mathbf{x} = (x_1, \cdots, x_N)$	$f(\boldsymbol{x})$
$\langle \mathbf{x}, \mathbf{y} \rangle$	$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(\boldsymbol{x})\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$
$\mathbf{e}_n$ (canonical basis)	Dirac distribution: $\delta \in \mathcal{S}'(\mathbb{R}^d)$
$\mathbf{y} = \mathbf{A}\mathbf{x}$	$\mathrm{A}\{f\}(\boldsymbol{x}) = \int_{\mathbb{R}^d} a(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$
Transpose of a matrix: $\mathbf{L}^T$	Adjoint operator: L^*

Normed spaces

Normed space: vector space $\mathcal{X}$ equipped with a norm $\|\cdot\|_{\mathcal{X}}$

Convergent sequence of functions (φ_i) in $\mathcal{X}$:

$$\lim_i \varphi_i = \varphi \quad \Leftrightarrow \quad \lim_i \|\varphi - \varphi_i\|_{\mathcal{X}} = 0$$

Definition
A Banach space is a **complete normed** space $\mathcal{X}$;
that is, such that $\lim_i \varphi_i = \varphi \in \mathcal{X}$ for any convergent sequence (φ_i) in $\mathcal{X}$.

- Lebesgue space $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$

$$\|\varphi\|_{L_p} = \begin{cases} \left(\int_{\mathbb{R}^d} |\varphi(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty \\ \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{R}^d} |\varphi(\boldsymbol{x})| & \text{for } p = \infty \end{cases}$$

- Space of continuous and bounded functions

$$C_{\mathrm{b}}(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{R} \text{ continuous and s.t. } \|f\|_{L_\infty} < +\infty \} \quad \subseteq L_\infty(\mathbb{R}^d)$$

Hilbert spaces

Definition A real-valued inner product on a vector space $\mathcal{H}$ is a bilinear form $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$: $(f, g) \mapsto \langle f, g \rangle_{\mathcal{H}}$ that satisfies the following properties for all $f, g, h \in \mathcal{H}$ and $\alpha \in \mathbb{R}$.

- Linearity: $\langle \alpha f, g \rangle_{\mathcal{H}} = \alpha \langle f, g \rangle_{\mathcal{H}}$ and $\langle f + g, h \rangle_{\mathcal{H}} = \langle f, h \rangle_{\mathcal{H}} + \langle g, h \rangle_{\mathcal{H}}$.
- Symmetry: $\langle f, g \rangle_{\mathcal{H}} = \langle g, f \rangle_{\mathcal{H}}$.
- Non-negativity: $\langle f, f \rangle_{\mathcal{H}} \geq 0$.
- Unicity: $\langle f, f \rangle_{\mathcal{H}} = 0 \Leftrightarrow f = 0$.

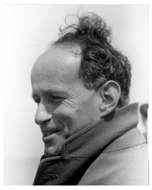
If all conditions except the last are met, then $\langle f, g \rangle_{\mathcal{H}}$ is called a semi-inner product.

Definition

A Hilbert space is a **complete normed** space $\mathcal{H}$ equipped with a norm induced by an inner product: $\|f\|_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}}$.

Example: $\langle f, g \rangle_{L_2} = \int_{\mathbb{R}^d} f(\boldsymbol{x}) g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$

Schwartz's space of test functions



Nuclear-Fréchet space (i.e., equipped with a family of semi-norms) rather than a Banach space

- $\mathcal{S}(\mathbb{R}^d)$: Schwartz' space of smooth and rapidly-decreasing functions

  Family of semi-norms: $\|\varphi\|_{\boldsymbol{m},\boldsymbol{n}} = \sup_{\boldsymbol{x} \in \mathbb{R}^d} |\boldsymbol{x}^{\boldsymbol{m}} \partial^{\boldsymbol{n}} \varphi(\boldsymbol{x})|$ for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^d$.

  $\lim_i \varphi_i = \varphi \quad \Leftrightarrow \quad \lim_i \|\varphi_i - \varphi\|_{\boldsymbol{m},\boldsymbol{n}} = 0$

$$\mathcal{S}(\mathbb{R}^d) = \left\{ \varphi : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } \|f\|_{\boldsymbol{m},\boldsymbol{n}} < +\infty, \text{ for all } \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^d \right\}$$

Very constrained (and safe) framework

$\mathcal{S}(\mathbb{R}^d) \subseteq L_p(\mathbb{R}^d)$ for any $p \geq 1$

Linear operators: continuity property

Definition
An operator $\mathrm{A} : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces is ***linear*** if, for any $\varphi_1, \varphi_2 \in \mathcal{X}$ and $a_1, a_2 \in \mathbb{R}$ (or $\mathbb{C}$),

$$\mathrm{A}\{a_1\varphi_1 + a_2\varphi_2\} = a_1\mathrm{A}\{\varphi_1\} + a_2\mathrm{A}\{\varphi_2\}$$

Definition
Let $\mathcal{X}, \mathcal{Y}$ be topological spaces. An operator $\mathrm{A} : \mathcal{X} \to \mathcal{Y}$ is (sequentially) ***continuous*** (with respect to the topologies of $\mathcal{X}$ and $\mathcal{Y}$) if, for any convergent sequence (φ_i) in $\mathcal{X}$ with limit $\varphi \in \mathcal{X}$, the sequence $\big(\mathrm{A}\varphi_i\big)$ converges to $\mathrm{A}\varphi$ in $\mathcal{Y}$, that is,

$$\lim_i \mathrm{A}\{\varphi_i\} = \mathrm{A}\{\lim_i \varphi_i\}.$$

Example: The linear operator D^{m_0} is continuous $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

Schwartz' semi-norms: $\|\varphi\|_{m,n} = \sup_{x\in\mathbb{R}} |x^m \mathrm{D}^n \varphi(x)|$ with $m, n \in \mathbb{N}$

$$\|\mathrm{D}^{m_0}\varphi_i - \mathrm{D}^{m_0}\varphi\|_{m,n} = \|\mathrm{D}^{m_0}\{\varphi_i - \varphi\}\|_{m,n} = \|\varphi_i - \varphi\|_{m,n+m_0} \to 0$$

Examples of continuous operators on $\mathcal{S}(\mathbb{R}^d)$

$\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$

- Multiplication by a polymonial: $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

$$\boldsymbol{r}^{\boldsymbol{n}}\varphi(\boldsymbol{r}) = r_1^{n_1}\cdots r_d^{n_d}\varphi(\boldsymbol{r}) \in \mathcal{S}(\mathbb{R}^d)$$

- Differentiation: $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

$$\partial^{\boldsymbol{n}}\varphi(\boldsymbol{r}) = \partial_{r_1}^{n_1}\cdots\partial_{r_d}^{n_d}\varphi(\boldsymbol{r}) \in \mathcal{S}(\mathbb{R}^d)$$

- Fourier transform: $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

$$\mathcal{F}\{\varphi\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{r}, \boldsymbol{\omega}\rangle}\varphi(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} \in \mathcal{S}(\mathbb{R}^d)$$

$$\mathcal{F}^{-1}\{\hat{\varphi}\}(\boldsymbol{r}) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{r}, \boldsymbol{\omega}\rangle}\hat{\varphi}(\boldsymbol{\omega})\,\frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^d} = \varphi(\boldsymbol{r})$$

Bounded operator

Proposition
Consider the linear operator $\mathrm{A} : \mathcal{X} \to \mathcal{Y}$ where $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are two Banach spaces. Then, A is continuous $\mathcal{X} \to \mathcal{Y}$ iff. it is bounded; i.e., iff. there exists a constant C_0 such that, for any $f \in \mathcal{X}$

$$\|\mathrm{A}\{f\}\|_{\mathcal{Y}} \le C_0 \|f\|_{\mathcal{X}}$$

- Examples
  - Boundedness of (classical) Fourier transform $\mathcal{F} : L_1(\mathbb{R}^d) \to C_{\mathrm{b}}(\mathbb{R}^d)$
  - Convolution operator: $\mathrm{T}_h\{f\} = h * f$

    T_h is bounded $L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \quad \Leftrightarrow \quad \hat{h} \in L_\infty(\mathbb{R}^d)$

$$\|h * f\|_{L_2}^2 = \int_{\mathbb{R}^d} \left|\hat{h}(\boldsymbol{\omega})\hat{f}(\boldsymbol{\omega})\right|^2 \frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^d} \le \|\hat{h}\|_{L_\infty}^2 \int_{\mathbb{R}^d} \left|\hat{f}(\boldsymbol{\omega})\right|^2 \frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^d} = \|\hat{h}\|_{L_\infty}^2 \|f\|_{L_2}^2$$

Spaces

$\mathscr{X}, \mathscr{Y}$	Generic vector spaces (normed or nuclear)
$L_2(\mathbb{R}^d)$	Finite-energy functions $\int_{\mathbb{R}^d} \lvert f(\boldsymbol{r})\rvert^2 \,\mathrm{d}\boldsymbol{r} < \infty$
$L_p(\mathbb{R}^d)$	Functions such that $\int_{\mathbb{R}^d} \lvert f(\boldsymbol{r})\rvert^p \,\mathrm{d}\boldsymbol{r} < \infty$
$L_{p,\alpha}(\mathbb{R}^d)$	Functions such that $\int_{\mathbb{R}^d} \left\lvert f(\boldsymbol{r})(1+\lvert\boldsymbol{r}\rvert)^\alpha\right\rvert^p \,\mathrm{d}\boldsymbol{r} < \infty$
$\mathscr{D}(\mathbb{R}^d)$	Smooth and compactly supported test functions
$\mathscr{D}'(\mathbb{R}^d)$	Distributions or generalized functions over $\mathbb{R}^d$
$\mathscr{S}(\mathbb{R}^d)$	Smooth and rapidly decreasing test functions
$\mathscr{S}'(\mathbb{R}^d)$	Tempered distributions (generalized functions)
$\mathscr{R}(\mathbb{R}^d)$	Bounded functions with rapid decay
$\ell_2(\mathbb{Z}^d)$	Finite-energy sequences $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \lvert a[\boldsymbol{k}]\rvert^2 < \infty$
$\ell_p(\mathbb{Z}^d)$	Sequences such that $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \lvert a[\boldsymbol{k}]\rvert^p < \infty$

Operators

Id	Identity
$\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$	Derivative
$\mathrm{D_d}$	Finite difference (discrete derivative)
D^N	Nth-order derivative
$\partial^{\boldsymbol{n}}$	Partial derivative of order $\boldsymbol{n} = (n_1, \dots, n_d)$
L	Whitening operator (LSI)
$\hat{L}(\boldsymbol{\omega})$	Frequency response of L (Fourier multiplier)

3.2 DUAL SPACES AND ADJOINT

- Intuition: finite dimensional case $\quad \forall \mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\forall a_1, a_2 \in \mathbb{R}$

$f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is a linear functional on $\mathbb{R}^n$

$\Leftrightarrow \quad f(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) = a_1 f(\mathbf{x}_1) + a_2 f(\mathbf{x}_2)$

$\Leftrightarrow \quad f(\mathbf{x}) = \mathbf{f}^T\mathbf{x} = \langle \mathbf{f}, \mathbf{x}\rangle$ for some $\mathbf{f} \in \mathbb{R}^n$

The set of all linear functionals on $\mathbb{R}^n$ is a vector space $(\mathbb{R}^n)^*$ isomorphic to $\mathbb{R}^n$

$$\mathbf{x} \mapsto \langle a_1\mathbf{f}_1 + a_2\mathbf{f}_2, \mathbf{x}\rangle = a_1\langle \mathbf{f}_1, \mathbf{x}\rangle + a_2\langle \mathbf{f}_2, \mathbf{x}\rangle$$

- Linear operator $\mathbf{y} \mapsto \mathbf{A}\mathbf{y}$: $\mathbb{R}^m \to \mathbb{R}^n$

$\mathbf{A}$: $n \times m$ matrix

$$\boxed{\langle \mathbf{f}, \mathbf{A}\mathbf{y}\rangle = \langle \mathbf{A}^*\mathbf{f}, \mathbf{y}\rangle}$$

$\mathbf{A}^* = \mathbf{A}^T$ is the adjoint of $\mathbf{A}$ $\qquad \mathbf{f} \mapsto \mathbf{A}^*\mathbf{f}$: $\mathbb{R}^n \to \mathbb{R}^m$

Algebraic vs. continuous duals

General vector space $\mathcal{X}$ (normed or nuclear)

Functional on $\mathcal{X}$: a map $\varphi \mapsto f(\varphi)$ that takes $\mathcal{X}$ to the scalar field $\mathbb{R}$

Notation for linear functionals: $f(\varphi) = \langle f, \varphi\rangle \qquad \forall \varphi \in \mathcal{X}$

The set of all linear functionals on $\mathcal{X}$ is a vector space $\mathcal{X}^*$ (algebraic dual of $\mathcal{X}$)

$$\langle a_1 f_1 + a_2 f_2, \varphi\rangle = a_1\langle f_1, \varphi\rangle + a_2\langle f_2, \varphi\rangle, \qquad \forall a_1, a_2 \in \mathbb{R}$$

The set of all ***continuous*** linear functionals on $\mathcal{X}$ is a vector space $\mathcal{X}' \subseteq \mathcal{X}^*$

$\mathcal{X}'$: topological or ***continuous dual*** of $\mathcal{X}$

Scalar (or duality) product $\langle\cdot,\cdot\rangle$ is a continuous bilinear functional: $\mathcal{X}' \times \mathcal{X} \to \mathbb{R}$

- Weak-$*$ topology on $\mathcal{X}'$: (f_i) converges to f in $\mathcal{X}'$

$$\Leftrightarrow \quad \lim_i \langle f_i, \varphi\rangle = \langle f, \varphi\rangle \text{ for all } \varphi \in \mathcal{X}$$

Dual of a Banach space

The dual of a Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is another Banach space $\mathcal{X}'$ equipped with the dual norm

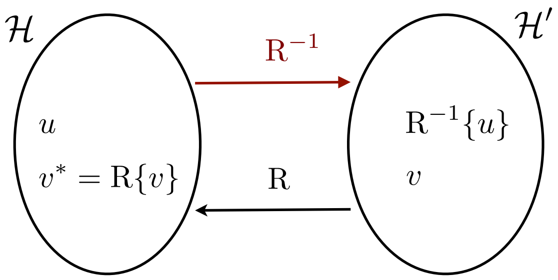
$$\|v\|_{\mathcal{X}'} = \sup_{u \in \mathcal{X}\setminus\{0\}} \left(\frac{\langle v, u\rangle}{\|u\|_{\mathcal{X}}} \right)$$

- Duals of L_p spaces $\quad \big(L_p(\mathbb{R}^d)\big)' = L_{p'}(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $p \in (1, \infty)$

Hölder inequality: $|\langle f, \varphi\rangle| \le \int_{\mathbb{R}^d} |f(\boldsymbol{r})\varphi(\boldsymbol{r})| \, \mathrm{d}\boldsymbol{r} \le \|f\|_{L_p} \|\varphi\|_{L_{p'}}$

$$\Rightarrow \quad \|f\|_{L_{p'}} = \sup_{\varphi \in L_p(\mathbb{R}^d)\setminus\{0\}} \left(\frac{\langle f, \varphi\rangle}{\|\varphi\|_{L_p}} \right)$$

Dual of a Hilbert space



Riesz isomorphism

$$\langle u, v^*\rangle_{\mathcal{H}} = \langle u, v\rangle = \langle \mathrm{R}^{-1}u, v\rangle_{\mathcal{H}'}$$

The dual of a Hilbert space $\mathcal{H}$ is a Hilbert space $\mathcal{H}'$ with $\mathcal{H}' = \mathrm{R}^{-1}(\mathcal{H})$.

Riesz' representation theorem
Let $(\mathcal{H}, \mathcal{H}')$ be a dual pair of Hilbert spaces. Then, for any $v \in \mathcal{H}'$, there is a unique element $v^* = \mathrm{R}\{v\} \in \mathcal{H}$ (the so-called **conjugate** of v) such that

$v(u) = \langle v^*, u\rangle_{\mathcal{H}}$ for all $u \in \mathcal{H}$.

Conversely, for any $v^* \in \mathcal{H}$, the linear functional $v : u \mapsto \langle v^*, u\rangle_{\mathcal{H}}$ is continuous with $\|v\| = \|v\|_{\mathcal{H}'} = \|v^*\|_{\mathcal{H}} = \|\mathrm{R}\{v\}\|_{\mathcal{H}}$, and hence included in $\mathcal{H}'$.

The linear isometric map $\mathrm{R} : \mathcal{H}' \to \mathcal{H}$ that associates any element $v \in \mathcal{H}'$ to its conjugate $v^* \in \mathcal{H}$ is called the **Riesz map**.

Dual of Schwartz' space of test functions

$\mathcal{S}'(\mathbb{R}^d)$: Schwartz's space of ***tempered distributions*** over $\mathbb{R}^d$

Generalized functions =

continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$

- Rigged Hilbert spaces: $\mathcal{H}_n(\mathbb{R}^d)$
  - matched order n of decay & smoothness:
    $f \in \mathcal{H}_n(\mathbb{R}^d) \quad \Leftrightarrow \quad (\cdot)^{\boldsymbol{m}}\partial^{\boldsymbol{n}} f \in L_2(\mathbb{R}^d),\ |\boldsymbol{m}|, |\boldsymbol{n}| \leq n$
  - $\mathcal{H}_{n_1}(\mathbb{R}^d) \subseteq \mathcal{H}_{n_2}(\mathbb{R}^d)$ for $n_1 \geq n_2$
  - $\big(\mathcal{H}_n(\mathbb{R}^d)\big)' = \mathcal{H}_{-n}(\mathbb{R}^d)$
  - Nuclear structure: $\mathcal{H}_{n+1} = \mathrm{T}\big(\mathcal{H}_n\big)$
    where T is a **nuclear** operator

$$\mathcal{S}'(\mathbb{R}^d) = \bigcup_{m\in\mathbb{Z}} \mathcal{H}_m(\mathbb{R}^d)$$

$\mathcal{H}_{-n}(\mathbb{R}^d)$

$\vdots$

$\mathcal{H}_0 = L_2(\mathbb{R}^d)$

$\vdots$

$\mathcal{H}_n(\mathbb{R}^d)$

$\vdots$

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{m\in\mathbb{Z}} \mathcal{H}_m(\mathbb{R}^d)$$

Adjoint operator

Pair of vector spaces $(\mathcal{X}, \mathcal{Y})$ with topological duals $(\mathcal{X}', \mathcal{Y}')$

- Linear operator $\varphi \mapsto \mathrm{A}\varphi$: $\mathcal{X} \to \mathcal{Y}$

$$\boxed{\underbrace{\langle f, \mathrm{A}\varphi\rangle}_{\mathcal{Y}' \times \mathcal{Y}} = \underbrace{\langle \mathrm{A}^* f, \varphi\rangle}_{\mathcal{X}' \times \mathcal{X}}}$$

$\mathrm{A}^* : \mathcal{Y}' \to \mathcal{X}'$ is the adjoint of A

Example: $\mathrm{D} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

$$\mathrm{D}^* = -\mathrm{D}$$

Integration by part

$$\langle \mathrm{D}\varphi, \phi\rangle = \int_{\mathbb{R}} \varphi'(x)\phi(x)\mathrm{d}x = \underbrace{\varphi(x)\phi(x)|_{-\infty}^{+\infty}}_{=0} - \int_{\mathbb{R}} \varphi(x)\phi'(x)\mathrm{d}x$$

Generalization: $(\partial^{\boldsymbol{n}})^* = (-1)^{|\boldsymbol{n}|}\partial^{\boldsymbol{n}}$ with $|\boldsymbol{n}| = n_1 + \cdots + n_d$

Hermitian inner product vs. duality product

L_2 (or hermitian) inner product: $\langle f, g\rangle_{L_2} = \int_{\mathbb{R}^d} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}$ $\quad f, g \in L_2(\mathbb{R}^d)$

Hermitian transpose: $\mathrm{A}^H = \overline{\mathrm{A}^*}$ $\quad \langle \mathrm{A}f, g\rangle_{L_2} = \langle f, \mathrm{A}^H g\rangle_{L_2} = \langle f, \overline{\mathrm{A}^* g}\rangle$

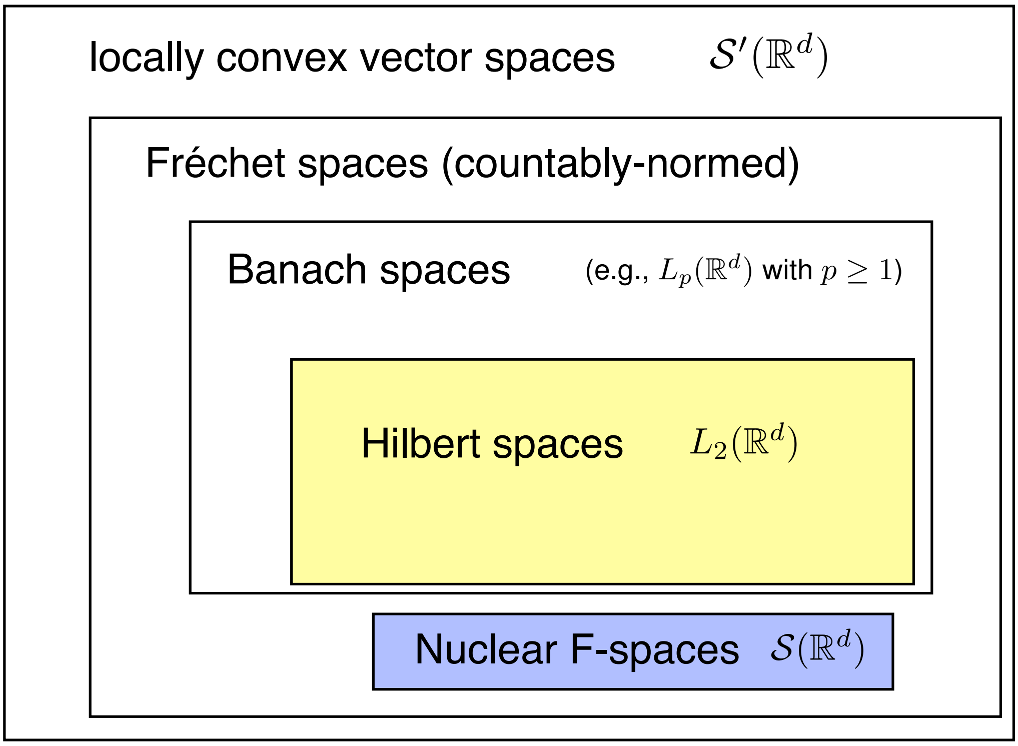
- Classical Fourier transform

$$\hat{f}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{r})\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{r}, \boldsymbol{\omega}\rangle}\, \mathrm{d}\boldsymbol{r}, \qquad f \in L_1(\mathbb{R}^d)$$

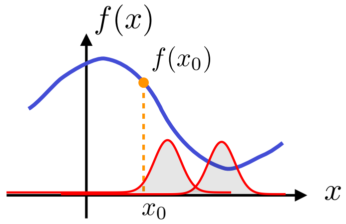
Plancherel's extension $\mathcal{F} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$

Parseval relation: $\langle f, g\rangle_{L_2} = \frac{1}{(2\pi)^d}\langle \hat{f}, \hat{g}\rangle_{L_2}$

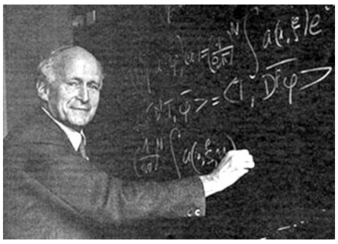
Duality product version: $\langle f, \hat{g}\rangle = \langle \hat{f}, g\rangle$



3.3 GENERALIZED FUNCTIONS



Physical measurement: $\langle \varphi, f\rangle = \int_{\mathbb{R}} \varphi(x) f(x) \mathrm{d}x$



Family of linear sensors $\langle \varphi_i, f\rangle$:

$$\langle a_1\varphi_1 + a_2\varphi_2, f\rangle = a_1\langle\varphi_1, f\rangle + a_2\langle\varphi_2, f\rangle$$

Continuity of the measurements with respect to variations in φ_i:

$$\lim_i \varphi_i = \varphi \quad \Rightarrow \quad \lim_i \langle \varphi_i, f\rangle = \langle \varphi, f\rangle$$

- Notion of weak equality

$$f = g \quad \Leftrightarrow \quad \langle\varphi, f\rangle = \langle \varphi, g\rangle \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

"If it looks like a duck, swims like a duck, and quacks like a duck, then it is (weakly) a duck"

What is a generalized function ?

A ***continuous linear functional*** on $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{D}(\mathbb{R}^d)$)

$f \in \mathcal{S}'(\mathbb{R}^d)$: Schwartz' space of tempered distributions

A rule $\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ that associates a real number $\langle f, \varphi\rangle$ to every test function φ

Examples: $\varphi \mapsto \langle \delta(\cdot - \boldsymbol{r}_0), \varphi\rangle = \varphi(\boldsymbol{r}_0)$

An extension of the classical notion of function.

If $g(\boldsymbol{x})$ is (slowly increasing and) locally integrable, then $\langle g, \varphi\rangle = \int_{\mathbb{R}^d} g(\boldsymbol{r})\varphi(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$

Otherwise the "integral" notation $\langle f, \varphi\rangle = \int_{\mathbb{R}^d} f(\boldsymbol{r})\varphi(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$ is only meant symbolically

Operations on generalized functions

Dual extension principle

Given operators $\mathrm{U}, \mathrm{U}^* : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ that form an adjoint pair on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$.
Their action to $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is extended by defining $\mathrm{U}f$ and $\mathrm{U}^* f$ such that

$$\langle \varphi, \mathrm{U}f \rangle = \langle \mathrm{U}^* \varphi, f \rangle,$$
$$\langle \varphi, \mathrm{U}^* f \rangle = \langle \mathrm{U}\varphi, f \rangle.$$

- Examples
  - Shift by $\boldsymbol{r}_0 \in \mathbb{R}^d$: $\langle \varphi, f(\cdot - \boldsymbol{r}_0) \rangle = \langle \varphi(\cdot + \boldsymbol{r}_0), f \rangle$
  - $\boldsymbol{n}$th-order derivative: $\langle \varphi, \partial^{\boldsymbol{n}} f \rangle = (-1)^{|\boldsymbol{n}|} \langle \partial^{\boldsymbol{n}} \varphi, f \rangle$
  - Convolution: $\langle \varphi, h * f \rangle = \langle h^\vee * \varphi, f \rangle$
  - Fourier transform: $\langle \varphi, \mathcal{F}\{f\} \rangle = \langle \mathcal{F}^*\{\varphi\}, f \rangle = \langle \hat{\varphi}, f \rangle$

Generalized Fourier transform

Definition

$\hat{f} = \mathcal{F}\{f\}$ is the ***generalized Fourier transform*** of $f \in \mathcal{S}'(\mathbb{R}^d)$ iff.

$$\langle \varphi, \hat{f} \rangle = \langle \hat{\varphi}, f \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $\hat{\varphi}(\boldsymbol{\omega}) = \mathcal{F}\{\varphi\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} \varphi(\boldsymbol{r}) \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{r}, \boldsymbol{\omega} \rangle} \, \mathrm{d}\boldsymbol{r}$.

- Fundamental property

The generalized Fourier transform is a reversible mapping $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$

- $f \in \mathcal{S}'(\mathbb{R}^d) \quad \Leftrightarrow \quad \mathcal{F}\{f\} \in \mathcal{S}'(\mathbb{R}^d)$
- $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}$

Examples of generalized Fourier transforms

- $\mathcal{F}\{\delta\} = 1$
- $\mathcal{F}\{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{\omega}_0, \boldsymbol{r} \rangle}\}(\boldsymbol{\omega}) = (2\pi)^d \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
- $\mathcal{F}\{|\boldsymbol{r}|^\gamma\}(\boldsymbol{\omega}) = C_\gamma \frac{1}{|\boldsymbol{\omega}|^{\gamma+d}}$

Temporal or spatial domain	Fourier domain
$\widehat{f}(\boldsymbol{r}) = \mathscr{F}\{f\}(\boldsymbol{r})$	$(2\pi)^d f(-\boldsymbol{\omega})$
$f^\vee(\boldsymbol{r}) = f(-\boldsymbol{r})$	$\hat{f}(-\boldsymbol{\omega}) = \hat{f}^\vee(\boldsymbol{\omega})$
$\overline{f(\boldsymbol{r})}$	$\overline{\hat{f}(-\boldsymbol{\omega})}$
$f(\mathbf{A}^{\mathrm{T}}\boldsymbol{r})$	$\frac{1}{\lvert\det \mathbf{A}\rvert}\widehat{f}(\mathbf{A}^{-1}\boldsymbol{\omega})$
$f(\boldsymbol{r}-\boldsymbol{r}_0)$	$\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{r}_0,\boldsymbol{\omega}\rangle}\widehat{f}(\boldsymbol{\omega})$
$\mathrm{e}^{\mathrm{j}\langle \boldsymbol{r},\boldsymbol{\omega}_0\rangle} f(\boldsymbol{r})$	$\widehat{f}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$
$\partial^{\boldsymbol{n}} f(\boldsymbol{r})$	$(\mathrm{j}\boldsymbol{\omega})^{\boldsymbol{n}}\widehat{f}(\boldsymbol{\omega})$
$\boldsymbol{r}^{\boldsymbol{n}} f(\boldsymbol{r})$	$\mathrm{j}^{\lvert\boldsymbol{n}\rvert}\partial^{\boldsymbol{n}}\widehat{f}(\boldsymbol{\omega})$
$(g * f)(\boldsymbol{r})$	$\widehat{g}(\boldsymbol{\omega})\widehat{f}(\boldsymbol{\omega})$
$g(\boldsymbol{r})f(\boldsymbol{r})$	$(2\pi)^{-d}(\widehat{g} * \widehat{f})(\boldsymbol{\omega})$

Table 3.3 Basic properties of the (generalized) Fourier transform.

Table A.1 Table of canonical regularizations of some singular functions, and their Fourier transforms. The one-sided power function is $r_+^\lambda = \frac{1}{2}\left(\lvert r\rvert^\lambda + \mathrm{sign}(r)\lvert r\rvert^\lambda\right)$ and Γ denotes the gamma function. Derivatives of δ are also included for completeness.

Singular function	Canonical regularization	Fourier transform
r_+^λ, $-n-1 < \mathrm{Re}(\lambda) < -n$	$\langle \varphi, \tilde{r}_+^\lambda\rangle = \int_0^\infty r^\lambda\left(\varphi(r) - \sum_{0\le i\le n-1}\frac{r^i\varphi^{(i)}(0)}{i!}\right)\mathrm{d}r$	$\frac{\Gamma(\lambda+1)}{(\mathrm{j}\omega)^{\lambda+1}}$
r_+^{-n}, $n = 1,2,3,\ldots$	$\langle \varphi, \tilde{r}_+^{-n}\rangle = \int_0^\infty r^{-n}\left(\varphi(r) - \left(\sum_{0\le i\le n-2}\frac{r^i\varphi^{(i)}(0)}{i!}\right) - \frac{r^{n-1}\varphi^{(n-1)}(0)}{i!}u(1-r)\right)\mathrm{d}r$ (non-canonical)	computable but not needed
r_+^n, $n = 0,1,2,\ldots$	N/A	$\mathrm{j}^n\pi\delta^{(n)}(\omega) + \frac{n!}{(\mathrm{j}\omega)^{n+1}}$
$\lvert r\rvert^\lambda$, $-2m-2 < \mathrm{Re}(\lambda) < -2m$	$\langle \varphi, \lvert\tilde{r}\rvert^\lambda\rangle = \int_0^\infty r^\lambda\left(\varphi(r) + \varphi(-r) - 2\sum_{0\le i\le m-1}\frac{r^{2i}\varphi^{(2i)}(0)}{(2i)!}\right)\mathrm{d}r$	$-2\sin(\frac{\pi}{2}\lambda)\frac{\Gamma(\lambda+1)}{\lvert\omega\rvert^{\lambda+1}}$
$\lvert r\rvert^\lambda\mathrm{sign}(r)$, $-2m-1 < \mathrm{Re}(\lambda) < -2m+1$	$\langle \varphi, \lvert\tilde{r}\rvert^\lambda\mathrm{sign}(r)\rangle = \int_0^\infty r^\lambda\left(\varphi(r) - \varphi(-r) - 2\sum_{0\le i\le m-1}\frac{r^{2i+1}\varphi^{(2i+1)}(0)}{(2i+1)!}\right)\mathrm{d}r$	$-2\mathrm{j}\cos(\frac{\pi}{2}\lambda)\frac{\Gamma(\lambda+1)}{\lvert\omega\rvert^{\lambda+1}}\mathrm{sign}(\omega)$
r^n, $n = 0,1,2,\ldots$	N/A	$\mathrm{j}^n 2\pi\delta^{(n)}(\omega)$
$r^n\mathrm{sign}(r)$, $n = 1,2,\ldots$	N/A	$2\frac{n!}{(\mathrm{j}\omega)^{n+1}}$
$1/r$,	$\int_0^{+\infty}\frac{\varphi(r)-\varphi(-x)}{r}\,\mathrm{d}r$	$-\mathrm{j}\pi\mathrm{sign}(\omega)$

The kernel theorem

Schwartz' kernel theorem: first form
Every continuous linear operator $\mathrm{A} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ can be written in the form

$$\varphi(\boldsymbol{r}) \mapsto \mathrm{A}\{\varphi\}(\boldsymbol{r}) = \int_{\mathbb{R}^d} a(\boldsymbol{r}, \boldsymbol{s})\varphi(\boldsymbol{s})\mathrm{d}\boldsymbol{s}$$

where $a(\cdot, \cdot)$ is a generalized function in $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$.

Generalized impulse response: $a(\cdot, \boldsymbol{s}_0) = \mathrm{A}\{\delta(\cdot - \boldsymbol{s}_0)\}$

Schwartz' kernel theorem: second form
Every continuous bilinear form $B : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ (or $\mathbb{C}$) can be written as

$$B(\varphi_1, \varphi_2) = \int_{\mathbb{R}^d \times \mathbb{R}^d} a(\boldsymbol{r}, \boldsymbol{s})\varphi_1(\boldsymbol{r})\varphi_2(\boldsymbol{s})\mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{r} = \langle \varphi_1, \mathrm{A}\{\varphi_2\}\rangle$$
$$= \langle a, \varphi_1 \otimes \varphi_2 \rangle$$

where $a(\cdot, \cdot)$ is a generalized function in $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$.

$(\varphi_1 \otimes \varphi_2)(\boldsymbol{r}, \boldsymbol{s}) = \varphi_1(\boldsymbol{r})\varphi_2(\boldsymbol{s})$ for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$

LSI operators and convolution

LSI: Linear Shift-Invariant

Shift-invariant operator:
$\mathrm{U}\{f(\cdot - \boldsymbol{r}_0)\}(\boldsymbol{r}) = \mathrm{U}\{f\}(\boldsymbol{r} - \boldsymbol{r}_0)$ for $\boldsymbol{r}_0 \in \mathbb{R}^d$

A continuous linear shift-invariant operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ can be written as a convolution

$$\varphi(\boldsymbol{r}) \mapsto (\varphi * h)(\boldsymbol{r}) = \int_{\mathbb{R}^d} \varphi(\boldsymbol{s})h(\boldsymbol{r} - \boldsymbol{s})\mathrm{d}\boldsymbol{s}$$

with some generalized function $h \in \mathcal{S}'(\mathbb{R}^d)$.

Kernel theorem with $a(\boldsymbol{r}, \boldsymbol{s}) = h(\boldsymbol{r} - \boldsymbol{s})$ (LSI property)

Fourier-domain multiplication: $\mathcal{F}\{h * \varphi\} = \hat{\varphi}\hat{h}$

- Special case: continuous operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

Smooth (and slowly increasing) Fourier multiplier: $\mathcal{F}\{h\}(\boldsymbol{\omega}) = \hat{h}(\boldsymbol{\omega})$

Example (rational transfer function): $\hat{h}(\omega) = C_0 \dfrac{\prod_{m=1}^{M}(\mathrm{j}\omega - z_m)}{\prod_{n=1}^{N}(\mathrm{j}\omega - p_n)}$

Convolution operators on $L_p(\mathbb{R}^d)$

- Young inequality for convolution

$\|h * f\|_{L_p} \leq \|h\|_{L_1} \|f\|_{L_p}$ for $p \geq 1$

$h \in L_1(\mathbb{R}^d)$ classical condition for BIBO stability

Definition
An operator $\mathrm{T} : L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$ is called a L_p ***Fourier multiplier*** if it is continuous on $L_p(\mathbb{R}^d)$ and can be represented as $\mathrm{T}f = \mathcal{F}^{-1}\{\hat{f}H\}$. The function $H : \mathbb{R}^d \rightarrow \mathbb{C}$ is the frequency response of the underlying filter.

Definition
The ***norm*** of the linear operator $\mathrm{T} : L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$ is given by

$$\|\mathrm{T}\|_{L_p} = \sup_{f \in L_p(\mathbb{R}^d)\setminus\{0\}} \frac{\|\mathrm{T}f\|_{L_p}}{\|f\|_{L_p}}.$$

The operator is said to be bounded if its norm is finite.

Characterization of Fourier multipliers

Theorem
Let T be a Fourier-multiplier operator with frequency response $H : \mathbb{R}^d \rightarrow \mathbb{C}$ and (generalized) impulse response $h = \mathcal{F}^{-1}\{H\} = \mathrm{T}\{\delta\}$. Then

1) The operator T is an L_1 Fourier multiplier if and only if there exists of a finite complex-valued Borel measure denoted by μ_h such that $H(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} \mu_h(\mathrm{d}\boldsymbol{x})$.

2) The operator T is an L_∞ Fourier multiplier if and only if H is the Fourier transform of a finite complex-valued Borel measure, as stated in 1).

3) The operator T is an L_2 Fourier multiplier if and only if $H = \hat{h} \in L_\infty(\mathbb{R}^d)$.

The corresponding operator norms are

$$\|\mathrm{T}\|_{L_1} = \|\mathrm{T}\|_{L_\infty} = \|\mu_h\|_{\mathrm{TV}} = \sup_{\|\varphi\|_{L_\infty} \leq 1} \langle \varphi, h \rangle$$

$$\|\mathrm{T}\|_{L_2} = \frac{1}{(2\pi)^{d/2}} \|H\|_{L_\infty},$$

where $\|\mu_h\|_{\mathrm{TV}}$ is the total variation of the underlying measure. Finally, T is an L_p Fourier multiplier for the whole range $1 \leq p \leq +\infty$ if the condition on H in 1) or 2) is met with $\|\mathrm{T}\|_{L_p} \leq \|\mu_h\|_{\mathrm{TV}}$.

finite-dimensional theory (linear algebra)	**infinite-dimensional theory (functional analysis)**
Euclidean space $\mathbb{R}^N$, complexification $\mathbb{C}^N$	function spaces such as the Lebesgue space $L_p(\mathbb{R}^d)$ and the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$, among others.
vector $\boldsymbol{x} = (x_1, \dots, x_N)$ in $\mathbb{R}^N$ or $\mathbb{C}^N$	function $f(\boldsymbol{r})$ in $\mathscr{S}'(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$, etc.
bilinear scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{n=1}^N x_n y_n$	$\langle \varphi, g \rangle = \int \varphi(\boldsymbol{r}) g(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$ $\varphi \in \mathscr{S}(\mathbb{R}^d)$ (test function), $g \in \mathscr{S}'(\mathbb{R}^d)$ (generalized function), or $\varphi \in L_p(\mathbb{R}^d)$, $g \in L_q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$, for instance.
equality: $\boldsymbol{x} = \boldsymbol{y} \iff x_n = y_n$ $\iff \quad \langle \boldsymbol{u}, \boldsymbol{x} \rangle = \langle \boldsymbol{u}, \boldsymbol{y} \rangle, \ \forall \boldsymbol{u} \in \mathbb{R}^N$ $\iff \quad \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 0$	various notions of equality (depends on the space), such as weak equality of distributions: $f = g \in \mathscr{S}'(\mathbb{R}^d) \iff \langle \varphi, f \rangle = \langle \varphi, g \rangle$ for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$, almost-everywhere equality: $f = g \in L_p(\mathbb{R}^d) \iff \int_{\mathbb{R}^d} \lvert f(\boldsymbol{r}) - g(\boldsymbol{r}) \rvert^p \, \mathrm{d}\boldsymbol{r} = 0$.
linear operators $\mathbb{R}^N \to \mathbb{R}^M$ $\boldsymbol{y} = \mathbf{A}\boldsymbol{x} \Rightarrow y_m = \sum_{n=1}^N a_{mn} x_n$	continuous linear operators $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ $g = \mathrm{A}\varphi \Rightarrow g(\boldsymbol{r}) = \int_{\mathbb{R}^d} a(\boldsymbol{r}, \boldsymbol{s}) \varphi(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}$ for some $a \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ (Schwartz' kernel theorem)
transpose $\langle \boldsymbol{x}, \mathbf{A}\boldsymbol{y} \rangle = \langle \mathbf{A}^{\mathrm{T}} \boldsymbol{x}, \boldsymbol{y} \rangle$	adjoint $\langle \varphi, \mathrm{A} g \rangle = \langle \mathrm{A}^* \varphi, g \rangle$

Table 3.1 Comparison of notions of linear algebra with those of functional analysis and the theory of distributions (generalized functions). See Sections 3.1-3.3 for an explanation.